Spectral triples for noncommutative solenoids

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Solenoids - nice compact spaces

Solenoid: a topological group Σ identified with an inverse limit of compact spaces $\varprojlim_{n\in\mathbb{N}} G_n$ with connecting maps $z \mapsto z^K$.
Starting from the subgroup of rationals, for $p \in \mathbb{N} \setminus \{0\}$,

$$\mathbb{Z}\left[\frac{1}{p}\right] = \left\{ \frac{a}{p^k} \in \mathbb{Q} \mid a \in \mathbb{Z}, k \in \mathbb{N} \right\},$$

write it as an inductive limit

$$\mathbb{Z} \xrightarrow{k\mapsto pk} \mathbb{Z} \xrightarrow{k\mapsto pk} \mathbb{Z} \to \cdots$$

and form its Pontryagin dual

$$\varprojlim \mathbb{T} = \mathbb{T} \xleftarrow{z\mapsto z^p} \mathbb{T} \xleftarrow{z\mapsto z^p} \mathbb{T} \longleftarrow$$

to get the p-solenoid

$$\mathcal{S}_p = \{(z_n)_n \in \mathbb{T}^{\mathbb{N}} \mid z_{n+1}^p = z_n, n \in \mathbb{N}\},$$

a compact group endowed with the topology inherited from the product $\mathbb{T}^{\mathbb{N}}$.

Noncommutative solenoids

Latrémolière and Packer (2013): Defined nc solenoids as twisted group C^*-algebras $C^*(\Gamma, \sigma_\theta)$ for a 2-cocycle that resembles the one on $\mathbb{Z}^2$ giving the rotation algebra $A_\theta = C^*(\mathbb{Z}^2, \sigma_\theta)$, with $\theta \in \mathbb{R}$.
Let p be a prime. Idea: replace $\mathbb{Z}^2$ with the group

$$\Gamma = \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$$

and pick the parameter θ in $\mathcal{S}_p$. Specifically, let Ω_p be

$$\{\theta = (\theta_n) \ \in \ \Pi_{n=0}^{\infty}[0,1)_n \mid \ \forall n \in \mathbb{N}, \ p\theta_n = \theta_{n-1} \bmod \mathbb{Z}\}.$$

The *noncommutative solenoid* $\mathcal{A}_\theta^{\mathcal{S}}$ is $C^*(\Gamma, \sigma_\theta)$ with multiplier

$$\left(\left(\frac{a_1}{p^{k_1}}, \frac{a_2}{p^{k_2}}\right), \left(\frac{a_3}{p^{k_3}}, \frac{a_4}{p^{k_4}}\right)\right) \xrightarrow{\sigma_\theta} e^{2\pi i \theta_{k_1+k_4} a_1 a_4}.$$

Noncommutative solenoids as nc spaces

1. Latrémolière and Packer, a series of papers (2015 onwards): the nc solenoids $\mathcal{A}_\theta^{\mathcal{S}}$'s have a structure of Leibniz quantum compact metric spaces and are limits in the Gromov-Hausdorff propinquity of noncommutative tori.
2. Austad-Luef (2021): a byproduct of their Gabor analysis is a description of a spectral triple on some nc solenoid.
3. Aiello-Guido-Isola a series of papers (2017 onwards): direct limit spectral triples which apply to periodic nc solenoids.
4. Enstad (2020): a Balian-Low theorem in the context of nc solenoids.
5. Lu (2022): a Morita equivalence study of nc solenoids

Questions: do (arbitrary) nc solenoids admit a fully-fledged theory of spectral triples?

Spectral triples for noncommutative solenoids

Our motivation and interest refer to:

1. Produce spectral triples on *all* noncommutative solenoids, including aperiodic ones (with simple C*-algebras), i.e. those where $\theta_j \neq \theta_k$ for $j \neq k$ in $\mathbb{N}$.
2. See if the spectral triple harmonizes with the inductive limit structure of $\mathcal{A}_\theta^{\mathcal{S}}$ as a limit of noncommutative tori, i.e. show that it is an inductive system (a'la Floricel-Ghorbanpour) of spectral triples on nc tori.
3. See if the spectral triple gives a Leibniz quantum compact metric space.
4. Show that there is a dense Fréchet $*$-subalgebra of $\mathcal{A}_\theta^{\mathcal{S}}$ that is stable under the holomorphic functional calculus.

Twisted group algebras

Let σ be any multiplier of the countable discrete group Γ, i.e., a 2-cocycle on Γ taking values in $\mathbb{T}$. For any $f_1, f_2 \in \ell^1(\Gamma)$, the twisted convolution $*_\sigma$ is given by

$$f_1 *_\sigma f_2 : \gamma \in \Gamma \mapsto \sum_{\gamma_1 \in \Gamma} f_1(\gamma_1) f_2((\gamma_1)^{-1}\gamma)\, \sigma(\gamma_1, (\gamma_1)^{-1}\gamma),$$

and the adjoint operation by

$$f_1^* : \gamma \in \Gamma \mapsto \overline{f_1(\gamma^{-1})\, \sigma(\gamma, \gamma^{-1})}.$$

Given a discrete group Γ and a multiplier σ on Γ, we define the **left-σ regular representation** λ_σ of the group algebra $\ell^1(\Gamma, \sigma)$ on $\ell^2(\Gamma)$ by, for all $f \in \ell^1(\Gamma, \sigma)$, $g \in \ell^2(\Gamma)$, γ_1, $\gamma \in \Gamma$:

$$\Big(\lambda_\sigma(f)(g)\Big)(\gamma) = \sum_{\gamma_1 \in \Gamma} \sigma(\gamma_1, \gamma_1^{-1}\gamma) f(\gamma_1) g(\gamma_1^{-1}\gamma).$$

Noncommutative solenoids as inductive limits

Fix the set of parameters

$$\Omega_p = \{\theta = (\theta_n) \in \Pi_{n=0}^{\infty}[0,1)_n \mid \forall n,\ p\theta_n = \theta_{n-1} \mod \mathbb{Z}\}.$$

Similar to A_θ's inductive limit realization as $\varinjlim C^*(\mathbb{Z}^2)$, the noncommutative solenoid $\mathcal{A}_\theta^{\mathcal{S}}$ has an inductive limit realization:

$$\varinjlim_{n\in\mathbb{N}} A_{\theta_{2n}} = A_{\theta_0} \xrightarrow{\varphi_0} A_{\theta_2} \xrightarrow{\varphi_1} A_{\theta_4} \xrightarrow{\varphi_2} \cdots A_{\theta_{2n}} \xrightarrow{\varphi_n} A_{\theta_{2n+2}} \cdots$$

To get this, restrict σ_θ to $\Gamma_n = \frac{1}{p^n}\mathbb{Z} \times \frac{1}{p^n}\mathbb{Z}$, subgroup of Γ, note that the generators $W_{(1/p^n,0)}$, $W_{(0,1/p^n)}$ of $\mathcal{A}_\theta^{\mathcal{S}}$ involve the even indices θ_{2n} in the parameter ($k_1 = k_4 = n$), and that

$$C^*\Big(\frac{1}{p^n}\mathbb{Z} \times \frac{1}{p^n}\mathbb{Z}, (\sigma_\theta)_n\Big) \cong A_{\theta_{2n}} \cong C^*(\mathbb{Z}^2,\ \sigma_{\theta_{2n}}).$$

Noncommutative solenoids as inductive limits

Recall

$$\Omega_p = \{\theta = (\theta_n) \ \in \ \Pi_{n=0}^{\infty}[0,1)_n \mid \ \forall n, \ p\theta_n = \theta_{n-1} \mod \mathbb{Z}\}.$$

Example (periodic rational case): cf. Aiello-Guido-Isola (2017). Let $p = 2$ and $\theta = (\frac{2}{3}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}, \cdots)$. This gives a noncommutative solenoid of periodic type

$$\mathcal{A}_\theta^{\mathcal{S}} = \varinjlim_n A_{\frac{2}{3}},$$

a direct limit of (copies of) the rational rotation algebra $A_{2/3}$. Aiello-Guido-Isola construct semifinite spectral triples on $\mathcal{A}_\theta^{\mathcal{S}}$.
Example (non-periodic rational case): p a prime, $a \in \mathbb{Z}$, $gcd(a, q) = 1$. We have $\theta = (\theta_n) \in \Omega_p$, and associated $\mathcal{A}_\theta^{\mathcal{S}}$, for

$$\theta_0 = \frac{a}{q}, \theta_1 = \frac{a}{pq}, \theta_2 = \frac{a}{p^2 q}, \ldots, \theta_{2n} = \frac{a}{p^{2n} q}, \ldots$$

Spectral triples

Following Connes, a **spectral triple** (A, H, D) consists of a unital C^*-algebra A, a unital faithful representation π of A on a Hilbert space H, and a self-adjoint operator $D : \mathrm{dom}(D) \subseteq H \to H$ such that

(ST1) the operator D has compact resolvent $R_\lambda(D) = (D - \lambda \mathrm{Id}_H)^{-1}$, $\lambda \in \mathbb{C} \setminus \mathrm{spec}(D)$,

(ST2) there exists a dense $*$-subalgebra $\mathcal{A}$, **smooth subalgebra** of A, such that for every $a \in \mathcal{A}$ the commutator

$$[D, \pi(a)] := D\pi(a) - \pi(a)D$$

is densely defined and extends to a bounded operator on H.

The classical case of $C(M)$ with H as the L^2-spinors on M recovers the geodesic distance by

$$d(x, y) = \sup\{|f(x) - f(y)| \mid f \in C(M), \|[D, f]\|_{B(H)} \leq 1\}.$$

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Spectral triples from length functions on groups

Definition (Connes, Rieffel)

A **length function** *on a discrete group* Γ *is* $\mathbb{L} : \Gamma \to [0, \infty)$ *such that*

1. $\mathbb{L}(\gamma) = 0$ *if and only if* $\gamma = e$*, where* e *is the identity of* Γ*,*
2. $\mathbb{L}(\gamma) = \mathbb{L}(\gamma^{-1})$ *for all* $\gamma \in \Gamma$*,*
3. $\mathbb{L}(\gamma_1\gamma_2) \leq \mathbb{L}(\gamma_1) + \mathbb{L}(\gamma_2)$ *for all* $\gamma_1, \gamma_2 \in \Gamma$*.*

A *Dirac operator* $D_{\mathbb{L}}$ on $C_c(\Gamma)$ (or $C_c(\Gamma, \sigma)$) associated to $\mathbb{L}$:

$$D_{\mathbb{L}}(f)(\gamma) = \mathbb{L}(\gamma) f(\gamma), \gamma \in \Gamma.$$

Candidate for a spectral triple (Γ usually finite generated)

$$\big(C^*(\Gamma, \sigma), \ell^2(\Gamma), D_{\mathbb{L}}\big).$$

Depending on $\mathbb{L}$, this could become a bona fide spectral triple.

Length functions on discrete groups

A length function $\mathbb{L} : \Gamma \to [0, \infty)$ is *proper* if every ball

$$B_{\mathbb{L}}(R) := \{\gamma \in \Gamma : \mathbb{L}(\gamma) \leq R\},$$

with $0 \leq R < \infty$ is finite, and has **bounded doubling** if

$$|B_{\mathbb{L}}(2R)| \leq C_{\mathbb{L}} \, |B_{\mathbb{L}}(R)|$$

for some finite constant $C_{\mathbb{L}}$. Alternatively, cf. Long-Wu (2021), $\mathbb{L}$ is of **bounded t-dilation** for a fixed $\mathbf{t} > 1$ if $\mathbb{L}$ is proper and

$$|B_{\mathbb{L}}(\mathbf{t}R)| \leq K_{\mathbb{L}} \, |B_{\mathbb{L}}(R)|$$

for some finite constant $K_{\mathbb{L}}$ and all $R \geq 1$.

Length functions on p-adic rationals

Proposition (Farsi-Landry-L-Packer)

Fix a prime p. Then $\mathbb{L}_p : \mathbb{Z}\left[\frac{1}{p}\right] \to [0, \infty)$ *given by*

$$\mathbb{L}_p(r) := |r| + \|r\|_p \text{ for } r \in \mathbb{Z}\left[\frac{1}{p}\right]$$

is a length function of bounded doubling with $C_{\mathbb{L}_p} = 4p^8$.

Consequently, $\mathbb{L} : \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right] \to [0, \infty)$ *given by*

$$\mathbb{L}(\gamma_1, \gamma_2) := \mathbb{L}_p(\gamma_1) + \mathbb{L}_p(\gamma_2)$$

is a length function of bounded doubling with $C_{\mathbb{L}} = (4p^8)^4$.

Key: the diagonal embedding $r \mapsto (r, -r)$ of $\mathbb{Z}[1/p]$ into $\mathbb{R} \times \mathbb{Q}_p$.

Spectral triple

Theorem (Farsi-Landry-L-Packer, 2022)

Fix a prime p. Let $\Gamma = \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$, $\theta \in \Omega_p$ *and* $C^*(\Gamma, \sigma_\theta)$ *with its left regular representation* λ_σ *on* $H = \ell^2(\Gamma)$*. Define* $\mathcal{D}_p$ *as the (unbounded) operator on* $H = \ell^2(\Gamma)$ *given by pointwise multiplication by* $\mathbb{L}$*. Then*

$$(\mathcal{A}_\theta^{\mathcal{S}}, H, \mathcal{D}_p)$$

with representation λ_σ *is a finitely summable spectral triple for the noncommutative solenoid* $\mathcal{A}_\theta^{\mathcal{S}}$ *with associated smooth subalgebra* $C_C(\Gamma, \sigma_\theta)$*.*

Length function of bounded doubling

About the proof: We use a length function on Γ with the property of *bounded doubling* or *bounded t-dilation*. Similar to arguments by Long-Wu, we have that the bounded doubling property implies that for $t > \log(C_{\mathbb{L}})$,

$$(\mathrm{Id}_H + \mathcal{D}_p^2)^{-t/2} \text{ is trace class;}$$

this requires convergence

$$\sum_{\gamma \in \Gamma} \frac{1}{(1 + \mathbb{L}(\gamma)^2)^{t/2}} < \infty.$$

The idea is to partition Γ and estimate the number of eigenvalues in distinct annuli, namely

$$\Gamma = B_{\mathbb{L}}(1) \sqcup \bigsqcup_{n=1}^{\infty} [B_{\mathbb{L}}(2^n) \setminus B_{\mathbb{L}}(2^{n-1})].$$

Noncommutative solenoids and spectral triples

Theorem (Aiello-Guido-Isola)

(2017) Let $\mathcal{A}_1$ be a C^*-algebra acted upon a finite abelian group G whose fixed-point algebra $\mathcal{A}_0$ is isomorphic to $\mathcal{A}_1$, and so that the eigenspaces of $\mathcal{A}_1$ under G contain invertible elements. Form the inductive limit $\varinjlim_n \mathcal{A}_n$, with $\mathcal{A}_n \cong \mathcal{A}_1$, seen as embedded in $\mathcal{A}_0 \otimes UHF(r^\infty)$, $r = |G|$.
There is a finitely summable, semifinite spectral triple with Dirac operator

$$D_0 \otimes I - 2\pi \sum_{a=1}^{2} \epsilon^a \otimes I \otimes (\sum_{j=1}^{\infty} I^{\otimes(j-1)} \otimes x_j^a),$$

with x_j^a acting diagonally and determined by sections s_j of $\widehat{\mathbb{Z}_B}$ into $A^j\mathbb{Z}^2$, for A, B certain matrices.

Spectral triples on general nc solenoids - I

Austad- Luef: spectral triples for noncommutative solenoids via Gabor analysis.
A Gabor system generated by $g \in L^2(\mathbb{R} \times \mathbb{Q}_p)$ and the lattice

$$\Lambda = \{(\alpha q, q, \beta r, r) \mid q, r \in \mathbb{Z}[1/p], \alpha, \beta > 0\}$$

is a family

$$\{\pi(\lambda)g\}_{\lambda \in \Lambda} = \{(t_\infty, t_p) \mapsto e^{2\pi i(\beta r t_\infty - \{r t_p\}_p)} g(t_\infty - \alpha q, t_p - q)\}$$

A Dirac operator can be defined as $\begin{pmatrix} 0 & f \\ f & 0 \end{pmatrix}$, with $f = v_s(x, \omega, q, r)$ determining a weighted Feichtinger algebra,

$$f = (1 + |x|^2 + |\omega|^2 + |q|^2 + |r|^2)^{s/2}, s \geq 0.$$

Spectral triples and inductive limits

Floricel-Ghorbanpour framework (2019): A morphism between two spectral triples (A_1, H_1, D_1) and (A_2, H_2, D_2) with respective smooth subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ is a pair (ϕ, I) of a unital *-homomorphism $\phi : A_1 \to A_2$ and a bounded linear operator $I : H_1 \to H_2$ with

1. $\phi(\mathcal{A}_1) \subseteq \mathcal{A}_2$,
2. $\pi_1(a) = \pi_2(\phi(a))I$, for every $a \in A_1$,
3. $I(\mathrm{dom}(D_1)) \subseteq \mathrm{dom}(D_2)$ and $ID_1 = D_2 I$.

For an inductive system of spectral triples

$$\{(A_j, H_j, D_j), (\phi_{j,k}, I_{j,k})_{j \leq k}\}_J,$$

an inductive realization consists of $A = \lim A_j$, $H = \lim H_j$, $\pi = \lim \pi_j$, $\mathcal{A} = \lim \mathcal{A}_j$ and D defined by $D\xi = I_j D_j \xi_j$ on $\xi = I_j \xi_j$, $\xi_j \in \mathrm{dom}(D_j)$. A priori, it *need not be a spectral triple*.

Spectral triples and inductive limits

Theorem (Floricel-Ghorbanpour (2019))

Given an inductive system of spectral triples

$$\{(A_j, H_j, D_j), (\phi_{j,k}, I_{j,k})_{j \le k}\}_J,$$

with inductive realization (A, H, D) *and* $\mathcal{A} = \lim \mathcal{A}_j$*, the following hold:*

(a) D *has compact resolvent iff the sequence* $\{I_j R_\lambda(D_j) I_j^*\}_{j\in\mathbb{N}}$ *converges uniformly to* $R_\lambda(D)$ *for some (every)* $\lambda \in \mathbb{C}\setminus\mathbb{R}$.

(b) The operator $[D, \pi(\phi_j(a))]$ *is bounded if and only if the family of operators* $\{[D_k, \pi_k(\phi_{j,k}(a))]\}_{k\ge j}$ *is uniformly bounded.*

Examples here: systems whose inductive realizations are spectral triples for AF-algebras (motivated by work of Christensen-Ivan).

Restriction to noncommutative tori

Proposition

Fix p, $\theta \in \Omega_p$, $n \in \mathbb{N}$. The restriction $\mathbb{L}_{p,n}$ of $\mathbb{L}_p$ to $\frac{1}{p^n}\mathbb{Z}$ is a length function of bounded doubling. For every $\theta \in \Omega_p$, let

$$\pi_{\theta_{2n}} : C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}}) \to B(\ell^2(\Gamma_n))$$

be the regular representation of $C^*(\Gamma_n, (\sigma_\theta)_n)$ followed by

$$C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}}) \cong C^*(\Gamma_n, (\sigma_\theta)_n).$$

The triple (obtained by restricting $\mathcal{D}_p$ to $\ell^2(\Gamma_n)$)

$$(C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}}), \ell^2(\Gamma_n), D_{p,n})$$

with representation $\pi_{\theta_{2n}}$ is a spectral triple for the noncommutative torus $C^*(\mathbb{Z}^2, \sigma_{\theta_{2n}})$, with smooth subalgebra $C_C(\mathbb{Z}^2, \sigma_{\theta_{2n}})$.

The spectral triple on $\mathcal{A}_\theta^{\mathcal{S}}$ as inductive limit

Theorem (Farsi-Landry-L-Packer)

Fix a prime p. Let $\Gamma = \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$ *and for each* $j \in \mathbb{N}$*, set* $\Gamma_j = \frac{1}{p^j}\mathbb{Z} \times \frac{1}{p^j}\mathbb{Z}$*. For every* $\theta \in \Omega_p$*, the triple* $(\mathcal{A}_\theta^{\mathcal{S}}, \ell^2(\Gamma), \mathcal{D}_p)$ *with smooth subalgebra* $C_C(\Gamma, \sigma_\theta)$ *can be written as the inductive realization of*

$$\{\, (C^*(\mathbb{Z}^2, \sigma_{\theta_{2j}}), \ell^2(\Gamma_j), D_{p,j}), (\phi_{j,k}, I_{j,k}) \,\}_{j\in\mathbb{N}},$$

each term with smooth subalgebra $(C_C(\mathbb{Z}^2, \sigma_{\theta_{2j}}))$*. Furthermore, the inductive realization* $(\mathcal{A}_\theta^{\mathcal{S}}, \ell^2(\Gamma), \mathcal{D}_p)$ *of the inductive system is itself a spectral triple, with compatible associated smooth subalgebras.*

Need $\{I_j(D_{p,j} - it)^{-1} I_j^*\}_{j\in\mathbb{N}}$ to converge in norm to $(\mathcal{D}_p - it)^{-1}$, and the family of commutators $\{\, [D_{p,k},\, \pi_{\theta_{2k}}(\phi_{J,k}(g))] \,\}_{k\geq J}$ to be uniformly bounded.

Quantum compact metric spaces

A **quantum compact metric space** (A, L) is an ordered pair where A is a unital C*-algebra and L is a seminorm defined on a dense $*$-subalgebra $\mathrm{dom}(L)$ of the self-adjoint elements A_{sa} such that:

(1) $\{a \in A_{sa} : L(a) = 0\} = \mathbb{R}1_A$,

(2) the Monge-Kantorovich metric mk_L, defined on the state space $\mathcal{S}(A)$ of A by setting for all $\varphi, \psi \in \mathcal{S}(A)$:

$$\mathrm{mk}_L(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathrm{dom}(L), L(a) \leq 1\}$$

metrizes the weak* topology restricted to the state space $\mathcal{S}(A)$ of A.

Such L on A is referred to as a **Lip-norm**.

Leibniz quantum compact metric spaces

A pair (A, L) with L being a Lip-norm on A is a **Leibniz quantum compact metric space** provided that L is lower semicontinuous wrt the norm topology restricted on its domain and, further, L satisfies

$$\max\left\{L\Big(\frac{ab+ba}{2}\Big), L\Big(\frac{ab-ba}{2i}\Big)\right\} \leq L(a)\,\|b\| + \|a\|\,L(b).$$

Relying on characterisations due to Long-Wu of Lip-norms on twisted group C*-algebras with length functions of bounded doubling (extending results of Christ-Rieffel), we have:

Theorem (Farsi-Landry-L-Packer)

For each prime p and $\theta \in \Omega_p$, the nc solenoid $\mathcal{A}_\theta^{\mathcal{S}}$ with $L_{\mathcal{D}_p}$ given by

$$L_{\mathcal{D}_p}(a) = \|[\mathcal{D}_p, \lambda_\sigma(a)]\|_{B(\ell^2(\Gamma))}$$

is a Leibniz quantum compact metric space.

Noncommutative solenoids - smooth subalgebras

Theorem (Austad, 2021)

Let G be a locally compact group and σ a multiplier on G. Assume that $L_1(G_c)$ is symmetric, with $G_c = G \times \mathbb{T}$ the Mackey group of C. Then $L^1(G, c)$ is symmetric. In particular, for any faithful representation $\pi : L^1(G, \sigma) \to B(H)$,

$$spec_{L^1(G,\sigma)}(f) = spec_{B(H)}(\pi(f)) \tag{1}$$

for each $f \in L^1(G, \sigma)$.

Combining this with results of Ludwig, we get the following:

Lemma

Let Γ be a countable discrete nilpotent group and σ a multiplier on Γ. Recall the left-σ regular representation on $\ell^1(\Gamma, \sigma)$. If $f \in \ell^1(\Gamma, \sigma)$, then

$$spec_{\ell^1(\Gamma,\sigma)}(f) = spec_{B(\ell^2(\Gamma))}(\lambda_\sigma(f)). \tag{2}$$

A Wiener's lemma for twisted group C^*-algebras

Extending Jolissaint's work to the twisted case, via results of Austad and Schweitzer, we obtain:

Theorem (Farsi-Landry-L-Packer)

Let Γ be a countable discrete nilpotent group and σ a multiplier on Γ. Suppose that $\mathbb{L}$ is a length function on Γ. Then the twisted Fréchet $*$-subalgebra $H^{1,\infty}_{\mathbb{L}}(\Gamma, \sigma)$ is dense and has the property of spectral invariance in $C^*(\Gamma, \sigma)$. Therefore, $H^{1,\infty}_{\mathbb{L}}(\Gamma, \sigma)$ is stable in $C^*(\Gamma, \sigma)$ under the holomorphic functional calculus.

Nc solenoids - fully fledged spectral triple

Theorem (Farsi-Landry-L-Packer)

Fix a prime p. Let $\Gamma = \mathbb{Z}\left[\frac{1}{p}\right] \times \mathbb{Z}\left[\frac{1}{p}\right]$. *For every* $\theta \in \Omega_p$, $(\mathcal{A}_\theta^{\mathcal{S}}, \ell^2(\Gamma), \mathcal{D}_p)$ *with representation* λ_{σ_θ} *is a spectral triple for the noncommutative solenoid* $C^*(\Gamma, \sigma_\theta) = \mathcal{A}_\theta^{\mathcal{S}}$, *with associated smooth subalgebra*

$$H^{1,\infty}_{\mathbb{L}}(\Gamma, \sigma_\theta) = \{f : \Gamma \to \mathbb{C} \mid \sum_\gamma |f(\gamma)(1 + \mathbb{L}(\gamma))| < \infty\}.$$

Furthermore, the twisted Fréchet $*$*-subalgebra* $H^{1,\infty}_{\mathbb{L}}(\Gamma, \sigma_\theta)$ *is a proper dense subalgebra of* $C^*(\Gamma, \sigma_\theta) = \mathcal{A}_\theta^{\mathcal{S}}$ *that is stable under the holomorphic functional calculus.*

THANK YOU.